$$(p^n+1)(p^{2n}-1)+1=p^n(p^{2n}+p^n-1)$$

distinct sets A, B, C, D for which $\triangle=0$.

Whether ω belongs to the $GF[p^n]$ or does not, the number of sets A, B, C, D making $\triangle=0$ is therefore $p^n(p^{2n}+p^n-1)$. It remains to determine for each set the number of elements I, γ for which f is not equal to 0, f_1 is not equal to 0. For p not equal to 2, this number is $(p^n-1)^2$ since $\mathrm{I}+\gamma$ and $\mathrm{I}-\gamma$ must take independently p^n-1 values. For $p=2$, $f=f_1$ and the number is $(2^n-1)2^n$.

The total number of sets I, α, ..., ε for which $D=0$ is therefore

$p^n(p^{2n}+p^n-1)(p^n-1)^2+p^{2n}(p^{3n}-p^{2n})+p^{5n}$ (if p is not equal to 2, p not equal to 3.)

$2^n(2^{2n}+2^n-1)(2^n-1)2^n+2^{5n}$ (if $p=2$).

Subtracting these numbers from p^{8n} and 2^{6n} respectively, we obtain the order of G. The results may be combined into the theorem:

The order of the group G in the $GF[p^n]$ is $p^n(p^n-1)^4(p^n+1)$ *if* $p>3$; $3^{4n}(3^n-1)^2$ *if* $p=3$; $2^{2n}(2^n-1)^3(2^n+1)$ *if* $p=2$.

Consider the group H the matrix of whose general transformation is derived from the matrix (1) by setting $\alpha=\beta$, $\gamma=\delta=\varepsilon$, the equal elements corresponding to the conjugate operators in the symmetric group g_8. The determinant D' of this special matrix is the special group-determinant of g_6. Then

$$(5) \qquad D'=(\mathrm{I}+2\alpha+3\gamma)(\mathrm{I}+2\alpha-3\gamma)(\mathrm{I}-\alpha)^4.$$

For $p=3$, $D'\equiv(\mathrm{I}-\alpha)^6$, so that the order of H is $3^{2n}(3^n-1)$.

For $p=2$, $D'\equiv(\mathrm{I}-\gamma)^2(\mathrm{I}-\alpha)^4$, so that the order of H is $2^n(2^n-1)^2$.

For $p>3$, we determine the number of sets I, α, γ in the $GF[p^n]$ for which $D'=0$. These are of three classes:

f' not equal to 0, f_1' not equal to 0, $\mathrm{I}-\alpha=0$;
f' not equal to 0, $f_1'=0$; $f'=0$,

where $f'=\mathrm{I}+2\alpha+3\gamma$, $f_1'=\mathrm{I}+2\alpha-3\gamma$. For the first class, $\mathrm{I}=\alpha$, $3\alpha\pm3\gamma$ is not equal to 0.

If $\alpha=0$, γ has p^n-1 values; if α is not equal to 0, γ has p^n-2 values. Hence

$$(p^n-1)+(p^n-1)(p^n-2)=(p^n-1)^2$$

is the number of sets in the first class. For the second class,

$$\mathrm{I}+2\alpha-3\gamma=0,\ \gamma \text{ is not equal to } 0,$$

giving $(p^n-1)p^n$ sets. The third class contains p^{2n} sets. Hence

$$(p^n-1)^2+(p^n-1)p^n+p^{2n}=3p^{2n}-3p^n+1$$

is the total number of sets making $D'=0$. Subtracting this number from p^{3n}, we obtain $(p^n-1)^3$ as the order of H for $p>3$.

The group* H is an invariant subgroup of G. In view of the preceding results, the order of the quotient-group is

$$p^n(p^{2n}-1) \text{ if } p>3 \text{ or if } p=2;\ 3^{2n}(3^n-1) \text{ if } p=3.$$

This result for p not equal to 3 is in accord with the general theory of group-matrices† by which the quotient-group is seen to be simply isomorphic with the group of binary substitutions of determinant unity in the $GF[p^n]$. The latter is known to be simple if $p=2$, and to have the factors of composition $\frac{1}{2}p^n(p^{2n}-1)$ and 2 if $p>2$.

The University of Chicago, March, 1902.

*The group H is evidently simply isomorphic with the commutative group of ternary linear transformations whose general matrix is

$$\begin{Bmatrix} 1 & 2\alpha & 3\gamma \\ \alpha & 1+\alpha & 3\gamma \\ \gamma & 2\gamma & 1+2\alpha \end{Bmatrix}$$

†Frobenius, Burnside, Dickson. See the references in the *Transactions of the American Mathematical Society*, July, 1902.

"THE BETWEENNESS ASSUMPTIONS."

By DR. ELIAKIM HASTINGS MOORE.

Amongst mathematicians there is abiding interest in the foundations of geometry—at present, in particular, as to the projective axioms. These axioms constitute, for instance, the first two groups I, II of Hilbert's system of axioms.

In a paper "On the Projective Axioms of Geometry," published January, 1902, in The Transactions of the American Mathematical Society (vol. 3, pp. 142-158), I exhibited and developed a new system of projective axioms for geometry of three or more dimensions, comparing it with the systems of Pasch, of Peano, and of Hilbert, and in this connection proving the redundancy in Hilbert's system I, II of the axioms I 4, II 4.

In the April 1902 number of THE MONTHLY (pp. 98-101), under the title, "The Betweenness Assumptions," Dr. Halsted published a second proof of the redundancy of II 4, a proof due to Mr. R. L. Moore, a student of his. Dr. Halsted alluded to my earlier proof of the theorem in the statement: "Mr. Moore has no intimation that any one has ever tried to prove these theorems." (l. c. p. 100.)

I wrote to Mr. Moore explaining the situation, and congratulating him upon the beauty of his proof. The congratulatory part of this letter appears in an editorial note (p. 148) of the May 1902 number of THE MONTHLY.

The letter was as follows:

THE UNIVERSITY OF CHICAGO, May 6, 1902.

MR. R. L. MOORE, The University of Texas, Austin, Texas.

MY DEAR MR. MOORE: I read with much interest, the other day, your proof of the redundancy of Hilbert's axiom II 4, in his system I, II, as exhibited by Professor Halsted in the current number of the AMERICAN MATHEMATICAL MONTHLY. Today I received from Professor Halsted a copy of that number. This is in response to a letter I sent him a week or so ago stating that I should be pleased to receive for publication in the Transactions the delightfully simple proof of the redundancy of which he wrote [had written] to me. I certainly agree with him in this estimate of your proof. Apparently he has not called your attention to the fact that the redundancy was pointed out by me and proved in my paper, which I am sending under separate cover, on the projective axioms of geometry, published in the January number of the Transactions. In accordance with correspondence with him, it was in connection with this paper of mine that he wrote to Hilbert and received Hilbert's response which led to your work on the subject. You will see that it was my desire to survey the whole system of projective axioms, and to exhibit a new system, and, in that connection to show that Hilbert's axioms I 4 and II 4 were in his system redundant, and, moreover, to furnish a satisfactory account of the rôles of the axioms I 3, 4, 5 which had been held by Schur to be redundant. As to the axiom II 4, you will see that, by considerations of the other linear axioms alone, and so in particular without the use of II 5, or of my axiom 4, I prove on page 151 that the axiom II 4 is a result of the statement 2_1, which [statement] is the statement of your theorem I. Thus to complete the proof of the redundancy of II 4, in Hilbert's system, I should today make use of your proof of theorem I. The proof that I give, in that it involves my triangle transversal axiom 4, is necessarily much longer.

I have supposed that you might be interested in understanding how your paper impresses me, and remain with considerable interest in the progress of your mathematical career,

Yours very truly,

(Signed) E. H. MOORE.

I suppose that the letter as a whole may be of value and interest to some readers of THE MONTHLY.

Chicago, June 3, 1902.

A NON-EUCLIDEAN GEM.

By DR. GEORGE BRUCE HALSTED.

La Géométrie non-euclidienne. Par P. Barbarin. Paris, C. Naud. 1902.

It is peculiarly appropriate that from Bordeaux, made sacred for non-Euclidean geometry by Hoüel, should emanate this beautiful little treatise, decor-

ated with a 'gravure' reproducing part of a manuscript of Euclid, also with the 'official' portrait of Lobachevski, but best of all with a portrait of Riemann.

It begins from the hackneyed position: "Experience therefore it is which has furnished to the ancient geometers a certrin number of primitive notions, of axioms, or fundamental postulates put by them at the basis of the science."

But now we know there never was any pure receptivity. In all thinking enters a creative element. Every bit of experience is in part created by the subject said to receive it, but really in great part making it.

Professor Barbarin continues: "From the epoch of Euclid, this number has been reduced to the strict minimum necessary, and all the others not comprised in this list, being capable of demonstration, are put in the class of theorems." Now we know that Euclid omits to notice many of the assumptions he unconsciously employs, for example all the "betweenness assumptions," while Hilbert has at last rigorously demonstrated Euclid's assumption, "All right angles are equal," and in turn one of Hilbert's assumptions has just been proven (see AMERICAN MATHEMATICAL MONTHLY, April, 1902, pp. 98-101).

The 'Elements' of Euclid, says Professor Barbarin, enjoyed throughout all the middle ages and still enjoy a celebrity that no other work of science has attained; this celebrity is due to their logical perfection, to the admirable concatenation of the propositions, and to the rigor of the demonstrations.

"Il mit dans son livre," says Montucla, "cet enchaînement si admiré par les amateurs de la rigueur géométrique." "In vain," he adds, "divers geometers whom this arrangement has displeased, have attempted to better it. Their vain efforts have made clear how difficult it is to substitute for the chain made by the Greek geometer another as firm and as solid." 'This opinion of the historian of mathematics,' says our author, 'retains all its value even after the researches which geometers have undertaken for about a century to submit the fundamental principles of the science to an acute and profound examination.'

I add that the remarkable discoveries of Dehn (see *Science*, N. S., Vol. XIV, pp. 711, 712), prove an unexpected superiority for Euclid over all successors down to our very day, and suggest the latest advance, which, though as yet unpublished, exists, for under date of April 2, 1902, Hilbert writes me: "In einer andern Arbeit will ich die Lobatschefski'sche Geometrie in der ebene unabhangig von Archimedes begrunden.' That is, Hilbert will found Bolyai's geometry as he has Euclid's, without any continuity assumption. To get the benefit of this brilliant achievement, I am holding back my own book on this fascinating subject.

Says Hilbert in his unpublished Vorlesung ueber Euklidische Geometrie, "The order of propositions is important. Mine differs strongly from that usual in text books of elementary geometry; on the other hand, it greatly agrees with Euclid's order.

"So fuehren uns diese ganz modernen Untersuchungen dazu, den Scharfsinn dieses alten Mathematikers recht zu wuerdigen und aufs hoechst zu bewundern."

Again, *apropos* of Euclid's renowned parallel postulate, Hilbert says: "What sagacity, what penetration the setting up of this axiom required we best recognize if we look at the history of the axiom of parallels. As to Euclid himself (circa 300 B. C.) he, *e. g.*, proves the theorem of the exterior angle before introducing the parallel axiom, a sign how deeply he had penetrated *in den Zusammenhang der geometrischen Saetze.*"

Professor Barbarin repeats the exploded error of attributing to Gauss the discovery of the non-Euclidean geometry in 1792. In the introduction to my translation of Bolyai's 'Science Absolute of Space,' pp. viii-ix, is a letter from Gauss, on which I there remark: "From this letter we clearly see that in 1799 Gauss was still trying to prove that Euclid's is the only non-contradictory system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false; the second can never be proven." In 1804 Gauss writes that in vain he still seeks the unloosing of this Gordian knot.

Again, with the date April 27, 1813, we read: "In the theory of parallels we are even now not farther than Euclid was. This is the *'partie hortense'* (shameful part) of mathematics, which soon or late must receive a wholly different form. Thus in 1813 there is in Göttingen still no light.

But in 1812 in Charkow, the non-Euclidean geometry already had been for the first time consciously created by Schweikart, whose summary characterization of it is given in *Science*, N. S., Vol. XII, pp. 842-846. This he communicated to Bessel and sent to Gerling and afterward to Gauss in 1818, so that it may claim to be the first *published* (not printed) treatise on non-Euclidean geometry. By this time Gauss had progressed far enough to be willing to signify *privately* his acceptance of Schweikart's doctrines.

On p. 15 Barbarin makes a brief argument for Euclid's axiom, "All right angles are equal." This argument was good before Hilbert and Veronese, since this axiom can never be proved by superposition. It is already a consequence of the assumptions preliminary to motion. This profounder analysis Barbarin has not attained to. He still uses as a postulate and supposes indispensable "l'indeformabilité des figures en déplacement." What Tules Andrade calls 'cette malheureuse et illogique définition' of Legendre, 'the shortest path between two points is a straight line,' Barbarin puts as an elementary proposition! Manning also, p. 2, assumes it, thus invalidating and making ephemeral his pretty little "Non-Euclidean Geometry" (Ginn & Co., 1901). Barbarin then proceeds to classify geometries by Saccheri's three hypotheses, the hypothesis of obtuse angle, the hypothesis of right angle, the hppothesis of acute angle, or that the angle sum of a rectilineal triangle is greater than, equal to, less than two right angles. But the remarkable discoveries of Dehn have now shown that this classification is invalid.

Barbarin says, p. 16, 'Saccheri proves that the hypothesis of the obtuse angle is incompatible with postulate 6' of Euclid. Dehn dissipates this supposed incompatibility by actually exhibiting a new geometry in which they amicably blend, which he calls the non-Legendrean geometry. In the same way,

the hypothesis of right angle amalgamates with the contradiction of Euclid's parallel-postulate in a geometry which Dehn calls semi-Euclidean. As Dehn states this result: There are non-Archimedian geometries in which the parallel-axiom is not valid and yet the angle-sum in every triangle is equal to two right angles. Thus the Theorem (Legendre, 12th Ed., I, 23; Barbarin, p. 25): 'If the sum of the angles of every triangle is equal to two right angles, the fifth postulate is true,' is seen to break down.

Manning's 'Non-Euclidean Geometry,' though it says (p. 93), 'The elliptic geometry was left to be discovered by Riemann,' gives only the single elliptic. It never even mentions the double elliptic, or spherical or Riemannian geometry, which Killing maintains was the only form which ever came before Riemann's mind. If so, then Barbarin's book is like Riemann's mind. The Riemannian, as distinguished from the single elliptic, is the only form which appears in it. Killing was the first, who (1879, *Crelle's Journal*, Bd. 83) made clear the difference between the Riemannian and the single elliptic space (or as he calls it, the polar form of the Riemannian).

Klein championed the single elliptic. Manning knows no other. Professor Simon Newcomb. like Manning, deals only with the single elliptic in his treatise; 'Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature *in the fourth dimension*.' The last four words F. S. Woods replaces by seven dots in his article 'Space of constant curvature' (*Annals of Math.*, Vol. 3, p. 72), though blaming Professor E. S. Crawley for the error they contain. Newcomb's also was the unfortunate statement which dubbed this "A Fairytale of Geometry,' a point of view from which he is still suffering in his latest little unburdening in *Harper's Magazine*. Just so Lobachevski had the misfortune to call his creation "Imaginary Geometry." Contrast John Bolyai's "The Science Absolute of Space."

In single elliptic space every complete straight line is of finite constant length πk. Every pair of straight lines intersect and return again to their point of intersection, but have no other point in common. In the so-called spherical space, that is the Riemannian space, two straight lines always meet in two points (opposites, or antipodal points,) which are πk from each other. The single elliptic makes the plane a unilateral or double surface, so that two antipodal points would correspond to one point, but to opposite sides of this one-sided plane with reference to surrounding three-dimensional elliptic space.

The geometry for two-dimensional Riemannian space coincides completely with pure spherics, that is, with spherics established from postulates which make no reference to anything off of the sphere, inside or outside the sphere. Hence the great desirability of a treatise on pure spherics. It would at the same time be true and available for Euclidean and for Riemannian geometry. Yet its relations to three-dimensional Euclidean and three-dimensional Riemannian space would differ radically.

Through every Riemannian straight line passes an infinity of planes also Riemannian, and in each of these this straight has a determined and distinct

center; but the straight is independent of the planes, and is defined by the postulates. Now in the sphere the great circle and the one *pseudo*-plane which contains and fixes it, namely the sphere, are inseparable, since any portion, however minute, of either determines all the other as well as its center and radius.

In the single elliptic geometry the elliptic straight line does not divide the elliptic plane into two separated regions. We can pass from any one point of the plane to any other point without crossing a given straight in it. Starting from the point of intersection of two straights and passing along one of them a certain finite length, we come to the intersection point again without having crossed the other straight. Hence we can pass from what seems one side of the straight line to what seems the other without crossing it, that is, it is uni-lateral or double.

This single elliptic geometry is never mentioned in Barbarin's book; just as the Riemannian is never mentioned in Manning's book. First take your choice, then buy your non-Euclidean geometry.

On p. 36, Barbarin gives to Gauss the honor which belongs to Wallis of being the first to remark that the existence of unequal similar figures is equivalent, in continuous space, to the parallel postulate.

In Chapter VII, 'Les Contradicteurs de la géométrie non Euclidienne,' Professor Barbarin makes with unanswerable vigor the argument which I gave in my 'Report on Progress in Non-Euclidean Geometry,' (*Science*, N. S., Vol. X, pp. 545-557). There I quoted Whitehead, who was the first to publish (March 10, 1898) "the extension of Bolyai's theorem by investigating the properties of the general class of surfaces in any non-Euclidean space, elliptic or hyperbolic, which are such that their geodesic geometry is that of straight lines in a Euclidean plane. "Such surfaces are proved to be real in elliptic as well as in hyperbolic space, and their general equations are found for the case when they are surfaces of revolution. In hyperbolic space, Bolyai's limit-surfaces are shown to be a particular case of such surfaces of revolution. The same principles would enable the problem to be solved of the discovery in any kind of space of surfaces with their 'geodesic' geometry identical with that of planes in any other kind of space."

Now not only the strikingly important problem solved by Whitehead, but also the analagous problem indicated, had both been solved by Barbarin and presented three months before to the Académie Royale de Belgique; but these investigations were only published after the appearance of my Report (October 20, 1899). They, as Barbarin says, p. 63, 'bring out in a striking manner the absolute independence of the three systems of geometry, which are able each to get everything from its own resources without need of borrowing anything from the others.' In each of the three spaces, Euclidean, Bolyaian, Riemannian, there exist surfaces whose geodesics have the metric properties of the straights of the two other spaces.

But the book in which these beautiful researches are published: 'Etudes de géometrie analytique non euclidienne par P. Barbarin, Bruxelles,' 1900, Hayez, pp. 168, has other titles to universal recognition.

Notwithstanding the ever-present example of Euclid, who never uses a construction or a figure which he has not introduced as following deductively from his two postulated figures, the straight and the circle, an insidious error crept into geometry, taught by Beman and Smith in the following words: (See their 'Geometry,' 1899, p. 70, § 112, or the AMERICAN MATHEMATICAL MONTHLY, Vol. IX, p. 131.)

"*Note on Assumed Constructions.* It has been assumed, up to prop. XXVIII, that all constructions were made as required for the theorems. Thus an equilateral triangle has been frequently mentioned, although the method of constructing one has not yet been indicated; a regular heptagon has been mentioned in ex. 93, and reference might be made to certain results following from the trisection of an angle, although the solutions of the problems, to construct a regular heptagon, and to trisect any angle, are impossible by elementary geometry. But the possibility of solving such problems has nothing to do with the logical sequence of the theorems."

This is a fundamental error. Thus for example, to construct an equilateral triangle on a given sect ['line segment,' they say,] requires and presumes the following lemma: "If A and B be any two given points, there is at least one point C whose sects from A and B are both congruent to the sect AB." This requires for its deduction a continuity assumption, such as that of Dedekind, Weierstrass, or Archimedes. But this totally changes the whole sequence of theorems. Hilbert establishes Euclidean geometry without any such assumption. His assumption of this construction would shatter his whole edifice. Moreover the fact that, even with a continuity assumption, in elliptic space it is not always, even then, possible to construct an equilateral triangle on a given base, shows that the assumption involves also the previous assumption that the straight line is not finite or closed, which again involves the betweenness assumptions so brilliantly reduced from five to four in the April number of the AMERICAN MATHEMATICAL MONTHLY, again involving the whole logical sequence of the theorems.

The construction so glibly assumed, to pass a circle through any three non-co-straight points, is equivalent to the assumption of the world-renowned parallel-postulate, and thus has everything in the world to do with the sequence of the theorems. The assumed construction of a triangle from three sects which are to be its sides, by the method of Beman and Smith, p. 76, is equivalent to the assumption of the Archimedes postulate, which again has everything to do with the logical sequence of the theorems. In fact just this assumption makes ephemeral the beautiful method of Saccheri used in the book we are reviewing.

Hence we can appreciate that astounding achievement of Bolyai's young genius, his § 34, where he solves for his universe, Eu., I, 31, To draw a straight line through a given point parallel to a given straight line. His brilliant lead was followed more than half a century later by Gerard, but it is Barbarin who has ended the matter by deducing from certain very simple constructions of the trirectangular quadrilateral all the fundamental plane constructions.

In chapter VIII ('La géométrie physique,' § 30, 'La forme géométrique de notre univers') our author stresses the idea, that even if our universe were exactly Euclidean, it would be forever impossible for us to demonstrate this. As I said in my 'Non-Euclidean Geometry for Teachers,' p. 14: "If in the mechanics of the world independent of man we were absolutely certain that all therein is Euclidean and only Euclidean, then Darwinism would be disproved by the reductio ad absurdum. All our measurements are finite and approximate only. The mechanics of actual bodies in what Cayley called the external space of our experience, might conceivably be shown by merely approximate measurements to be non-Euclidean, just as a body might be shown to weigh more than two grams or less than two grams, though it never could be shown to weigh precisely, absolutely two grams."

Our author suggests the following experiment for proving our space non-Euclidean: From a point trace six rays sixty degrees apart. On them successively mark off the sects OA_0, OA_1, OA_2, ..., OA_n, of which each is the projection of the following. If we finish by finding between OA_n and 2^nOA_0 a difference of constant sense and greater than imputable to error of procedure, our universe is non-Euclidean.

In conclusion this beautiful little book has the advantage of being the production of an active and fertile original worker in the domain of which it treats. His 'Géométrie general des espaces' (1898), his 'Sur le paramétre de l' univers,' and 'Sur la géométrie des êtres plans' (1901), 'Le cinquiéme livre de la métagéométrie,' (1901), 'Les cosegments et les volumes en géométrie non euclidienne' (1902), and his 'Poligones réguliers spheriques et non-euclidiens,' shortly to appear in that virile young monthly, *Le Matematiche*, and which I had the advantage of reading in manuscript, show that Bordeaux is honored by a worthy successor of Hoüel, so universally beloved.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

Remark on Problem No. 154. by G. B. M. Zerr.

According to the correction made in the April number, the result in the second solution is 17. This result is not correct. The problem as stated is not possible. It takes 12 oxen to eat the growing grass. Then $17-12=5$ oxen remaining to eat the grass already grown.

Now 9 oxen eat the standing grass in 6 weeks or 1 ox eats it in 54 weeks.

$\therefore$ 5 oxen will eat it in $54\div5=10\frac{4}{5}$ weeks.

$\therefore$ 17 oxen will eat it, the grass, together with what grows, in $10\frac{4}{5}$ weeks.

157. Proposed by B.F.FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

January 1, 1899, A and B entered into partnership for 3 years. A put in \$10,000 and B put in \$5,500. July 1, 1899, B put in \$1,500 more. October 1, A took out \$500. January 1, 1900, each put in \$1,500. July 1, 1900, they dissolved partnership, and found that they had lost \$846. What is each partner's share of the loss?

Solution by J. R. HITT, Choral Institute, San Marcos, Texas, and HON. JOSIAH H. DRUMMOND, Portland, Maine.

A has in \$10,000 for 9 months, \$9,500 for 3 months, \$11,000 for 6 months.

B has in \$5,500 for 6 months, \$7,000 for 6 months, \$8,500 for 6 months.

Assuming the loss to be 10% of investment, A's loss would be \$750+\$237.50+\$550=\$1537.50. B's would be \$275+\$350+\$425=\$1050.

Hence, \$1537.50+\$1050 : \$846=\$1537.50 : \$502.696, A's loss.

\$1537.50+\$1050 : \$846=\$1050 : \$343.304, B's loss.

Also solved by *G. B. M. ZERR*. Professor Hitt should have received credit for solving 156.

158. Proposed by JAMES F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

My agent sold pork at 5% commission; increasing the proceeds by \$20, I ordered the purchase of flour at 3% commission; after which flour rose 9%, my whole gain was \$40. What did he sell the pork for?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let 100%=selling price of pork.

$100\% + \$20$=total cost.

$(95\% + \$20)\frac{109}{103} = 100\frac{55}{103}\% + \$21\frac{17}{103}$, selling price of flour.

$100\frac{55}{103}\% + \$21\frac{17}{103} - 100\% - \$20 = \$40$ gain.

$\therefore \frac{55}{103}\% = \$38\frac{86}{103}$. $\therefore 1\% = \$72\frac{8}{11}$.

$100\% = \$7272\frac{8}{11}$, selling price of pork.

Also solved by *J. R. HITT*.

ALGEBRA.

145. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Factorize $2b^2c^2+2c^2a^2+2a^2b^2+2a^2d^2+2b^2d^2+2c^2d^2-a^4-b^4-c^4-d^4$.

No correct solution of this problem has been received.

146. Proposed by B. F. YANNEY, Professor of Mathematics, Mount Union College, Alliance, Ohio.

If the series 1, 3, 5, ... $2n-1$, ... be divided into successive groups of r terms each, the sum of the terms of the nth group will be $(2n-1)$ times the sum of the terms of the first group, or $(2n-1)r^2$.

Solved by H. S. VANDIVER, Bala, Pa., and E. D. GRABER, Professor of Mathematics, Geneseo State Normal School, Geneseo, N. Y.

The n groups in question are

$$\begin{array}{c} 1, 3, 5, \ldots. 2r-1 \\ 2r+1, 2r+3, \ldots. 4r-1 \\ 4r+1, 4r+3, \ldots. 6r-1 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ 2(n-1)r+1, \ldots. 2nr-1 \end{array}$$

The sum of the terms in the nth group is

$$\frac{r}{2}(\text{1st term}+\text{last term})=\frac{r}{2}(4rn-2r)=r^2(2n-1).$$

Solved similarly by *G. B. M. ZERR, J. H. DRUMMOND,* and *J. SCHEFFER.*

147. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, Gloucestershire, England.

Prove that $x=a^x$ has never more than two real roots, and find the condition for no real roots.

No solution of this problem has been received.

148. Proposed by R. D. BOHANNAN, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\beta}+\frac{z}{c+\gamma}=1$, $\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=1$, $\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1$, show, without solving, that $x+y+z=a+\alpha+b+\beta+c+\gamma$.

No solution of this problem has been received.

149. Proposed by JOSEPH V. COLLINS, Ph. D., Stevens Point, Wis.

1. How many different football elevens can be sent out from a school having twenty players? In how many ways can eleven men line up?

Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

It is possible to send out $\frac{20!}{11!\ 9!}$ elevens, or 167960 elevens.

The eleven men can line up 11! ways.

Also solved by *G. B. M. ZERR* and *C. A. LINDEMANN.*

150. Proposed by JOSEPH V. COLLINS, Ph. D., Stevens Point, Wis.

2. How many sets of officers (president, vice-president, treasurer, and secretary) can a society of forty persons elect? How many committees of four persons, supposing no attention is paid to positions on the committees? How many committees in which the chairman is selected?

Solution by P. H. PHILBRICK, C. E., Lake Charles, La., and C. A. LINDEMAN, Professor of Mathematics Virginia Union University, Richmond, Va.

The society can elect $\frac{40!}{36!.4!}$ sets of officers.

The number of committees, no attention being paid to positions on the

same, is also $\frac{40!}{36!.4!}=\frac{40.39.38.37}{1.2.3.4}$.

The number of committees in which the chairman is selected, leaving 39 from whom to choose, is $\frac{39!}{36!.3!}=\frac{39.38.37}{6}=9139$.

Also solved by *G. B. M. ZERR.*

151. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve the equations:

$$x+y+z+u+w=1.$$
$$ax+by+cz+du+ew=h,$$
$$a^2x+b^2y+c^2z+d^2u+e^2w=h^2,$$
$$a^3x+b^3y+c^3z+d^3u+e^3w=h^3,$$
$$a^4x+b^4y+c^4z+d^4u+e^4w=h^4.$$

Solution by CLARENCE E. COMSTOCK, Professor of Mathematics, Bradley Polytechnic Institute, Peoria, Ill.

Solution by determinants.

$$x=\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ h & b & c & d & e \\ h^2 & b^2 & c^2 & d^2 & e^2 \\ h^3 & b^3 & c^3 & d^3 & e^3 \\ h^4 & b^4 & c^4 & d^4 & e^4 \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \\ a^2 & b^2 & c^2 & d^2 & e^2 \\ a^3 & b^3 & c^3 & d^3 & e^3 \\ a^4 & b^4 & c^4 & d^4 & e^4 \end{vmatrix} \equiv \frac{\Delta_{a=b}}{\Delta}.$$

By the factor theorem, we get

$$\Delta=(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e).$$

$\Delta_{a=h}$=the same with a replaced by h.

$$\therefore x=\frac{(h-b)(h-c)(h-d)(h-e)}{(a-b)(a-c)(a-d)(a-e)}.$$

Since a, b, c, d, e appear in the same way, the principle of symmetry enables us to write the values y, z, u, w at once.

$$y=\frac{(h-a)(h-c)(h-d)(h-e)}{(b-a)(b-c)(b-d)(b-e)}, z=\frac{(a-h)(c-h)(d-h)(e-h)}{(b-a)(c-a)(d-a)(e-a)},$$

$$u=\frac{(a-h)(b-h)(c-h)(e-h)}{(a-d)(b-d)(c-d)(e-d)}, \text{ and } w=\frac{(a-h)(b-h)(c-h)(d-h)}{(a-e)(b-e)(c-e)(d-e)}.$$

Solved in a similar manner by *G. B. M. ZERR.*

152. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Solve by a short original method, if possible:

$$x/a+y/b+c/z=P....(1),$$
$$x/a+b/y+z/z=Q....(2),$$
$$a/x+y/b+z/z=R....(3).$$

Solution by the PROPOSER.

Let $x/a=u$, $y/b=v$, $z/c=w$.

$u+v+1/w=P....(1)$, $u+1/v+w=Q....(2)$, $1/u+v+w=R....(3)$.

(1)$-$(2) gives $v^2w+v-vw^2-Pvw+Qvw=w....(4)$.

(1)$\times$(3) gives $1=(P-v-1/w)(R-v-w)$, or

$$v^2w+v+vw^2-Pvw-Rvw=Pw^2-PRw+R....(5).$$

(5)$-$(4) gives $vw=\dfrac{(w-R)(Pw-1)}{2w-R-Q}....(6)$.

(5)$\div$(4) gives $\dfrac{vw+1+w^2-Pw-Rw}{vw+1-w^2-Pw+Qw}=\dfrac{Pw^2-PRw+R}{w}....(7)$.

(6) in (7) gives

$$\frac{2w^3-(P+Q+3R)w^2+(1+R^2+PQ+RQ)w-Q}{-2w^3+(R-P+3Q)w^2+(1-Q^2+PQ-RQ)w-Q}=\frac{Pw^2-PRw+R}{w}.$$

$\therefore 2Pw^5+(2-3PR-3PQ+P^2)w^4+(4PQR+PR^2-P^2R+PQ^2-P^2Q-2P-Q-R)w^3+(1+2PQ-2QR+2PR-PQ^2R+P^2QR-PR^2Q)w^2+(RQ^2+R^2Q-R-Q-2PQR)w+RQ=0.$

This is an equation of the 5th degree and I have thus far been unable to solve it.

Let $P=Q=R$. Then

$$2Pw^5+(2-5P^2)w^4+4P(P^2-1)w^3+(1+2P^2-P^4)w^2-2Pw+P^2=0.$$

Let $P=3$.

$\therefore w^5-\frac{43}{6}w^4+16w^3-\frac{31}{3}w^2-w+\frac{3}{2}=0$, or $(w-\frac{1}{2})(w-1)(w-3)(w-3)(w-\frac{1}{3})=0$.

$\therefore$ w$=\frac{1}{2}$, 1, 3, 3 or $-\frac{1}{3}$; $v=\frac{1}{2}$, 1, 3, $-\frac{1}{3}$ or 3; $u=\frac{1}{2}$, 1, $-\frac{1}{3}$, 3 or 3.

GEOMETRY.

176. Proposed by R. A. WELLS, Franklin College, New Athens, Ohio.

If there be three straight lines which meet in a point, and the arbitrary constants of their equations, expressed in the slope form, be taken as the co-ordinates of three points, these three points will lie in a straight line.

Solution by ANNA L. VAN BEUSCHOTEN, Professor of Mathematics, Wells College, Aurora, N. Y.

Let the three straight lines be given by the equation

$$y=ax+b,\ y=cx+d,\ y=ex+f.$$

The condition that these lines intersect in a common point is given by the vanishing of the determinant,

$$\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

But the vanishing of this determinant is also the condition that the points (a, b), (c, d), (e, f) are colinear.

Also solved by *G. B. M. ZERR*.

177. Proposed by GEORGE LILLEY, Ph. D., LL. D., University of Oregon, Eugene, Ore.

If two medians of a triangle intersect each other at right angles, the third median will be the hypotenuse of a right triangle, of which the other two will be the sides.

Solution by H. B. PENHOLLOW, DeWitt Clinton High School, New York, N. Y.

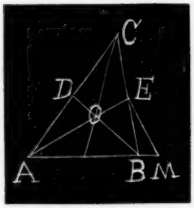
Given $\triangle ABC$, medians meeting at O, having $\angle AOB$ a right angle.

From E draw EM perpendicular to AE, meeting AB produced in M. Then $\triangle AEM$ is a right triangle in which AE is one median, $EM=DB$ another median. Also since triangles AEM and AOB are similar, $AE/AO=AM/AB$. But $AE=\frac{3}{2}AO$.

$\therefore AM=\frac{3}{2}AB$.

Also OF is median of right triangle AOB.

$\therefore OF=\frac{1}{2}AB$, or $CF=\frac{3}{2}AB=AM$. Q. E. D.



Also solved by *P. S. BERG, HENRY HEATON, P. H. PHILBRICK, C. A. LINDEMANN, G. I. HOPKINS, S. E. HARWOOD, J. F. LAWRENCE, T. T. DAVIS, G. B. M. ZERR*, and *ANNA BENCHOTEN*. Professor Penhollow and Miss Benchoten each furnished three solutions.

178. Proposed by JOHN M. ARNOLD, Crompton, R. I.

A cylinder thirty feet long and two feet in diameter is to be placed in a machinery car, the inside dimensions of which are eight feet wide and eight feet high. Find length of the shortest car that will contain it.

No correct solution of this problem has been received.

179. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University of Mississippi.

Of all isosceles triangles inscribed in a circle, the equilateral is the maximum and has the maximum perimeter. Prove geometrically.

Solution by the PROPOSER.

Case I. (See Fig. 1.) Vertical angle of isosceles triangle less than 60°.

Let ABC be an inscribed equilateral triangle and ADE any inscribed isosceles triangle with its base DE parallel to BC.

1st. To prove $\triangle ABC$ greater than $\triangle ADE$. $\triangle AFG$ is common.

Through D draw a parallel to EA meeting BC at H, and forming the parallelogram $DHGE$. It is easily proved that H always lies to the right of B.

Hence, HG is less than BG. Draw the diagonal HE.

Comparing triangles HGE and AGC,

$$\triangle HGE : \triangle AGC = HG.GE : AG.GC.$$

In $\triangle BAG$, BG is less than AG, and therefore, HG is less than AG. Also, in $\triangle GEC$, GE is less than GC.

Therefore, $\triangle HGE$ is less than $\triangle AGC$.

Hence parallelogram $HGED$ is less than $\triangle AGC + \triangle AFB$, and trapezoid $DFGE$, being a part of this parallelogram, is less than the sum of the same two triangles.

Therefore $\triangle ABC$ is greater than $\triangle ADE$.

2nd. To prove the perimeter of $\triangle ABC$ greater than that of $\triangle ADE$.

Draw FJ and GI perpendicular to AF and AG, respectively.

Draw CK perpendicular to GI produced, and FM and GL perpendicular to DE. AI is greater than AG, AJ is greater than AF, $FG = ML$.

In similar triangles CGK and GLE, CG is greater than EG, CK (and, therefore, CI) is greater than EL.

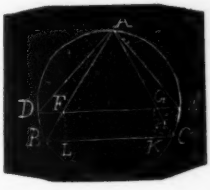
Similarly, BF is greater than DF, and BJ is greater than DM.

It follows that the perimeter of $\triangle ABC$ is greater than that of $\triangle ADE$.

Case II. (See Fig. 2.) Vertical angle of isosceles triangle greater than 60°.

Let ABC be an inscribed equilateral triangle and ADE an inscribed isosceles triangle with its base, DE, parallel to BC.

1st. To prove $\triangle ABC$ greater than $\triangle ADE$.



$\triangle AFG$ is common. Draw FC.

$\triangle FGC$ is greater than $\triangle AGE$, since $FG = AG$, and in $\triangle GCE$, GC is greater than GE.

It follows that trapezoid $BFGC$ is greater than the sum of triangles AGE and AFD.

Hence $\triangle ABC$ is greater than $\triangle ADE$.

2nd. To prove perimeter of $\triangle ABC$ greater than that of $\triangle ADE$.

On AC lay off AH equal to AE. Connect H and E.

$\angle AEH = \angle AHE$. $\angle GEH = \angle AEH - \angle AEG$.

Also, $\angle CEH = \angle AHE - \angle HCE$. But $\angle HCE = \angle AEG$.

Therefore, $\angle GEH = \angle CEH$, or HE is the bisector of $\angle GEC$.

Therefore $CH : HG = CE : GE$.

But CE is greater than GE.

Therefore CH is greater than HG, or CH is greater than one-half of CG.

Draw GK and FL perpendicular to BC. $KC = \frac{1}{2}GC$.

Adding, $KC + CH$ is greater than CG. But GC is greater than GE.

Therefore $KC + CH$ is greater than GE.

Hence $AC + CK$ is greater than $AE + EG$.

Similarly, $AB+BL$ is greater than $AD+DF$. And, since $LK=FG$, the perimeter of $\triangle ABC$ is greater than the perimeter of $\triangle ADE$.

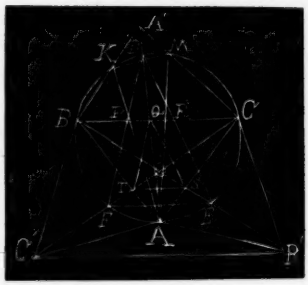
Excellent demonstrations were received from *P. H. PHILBRICK, G. B. M. ZERR, HENRY HEATON, C. A. LINDEMAN*, and *T. T. DAVIS*.

180. Proposed by R. TUCKER, M. A.

ABC is a triangle; A', B', C' are the images of A, B, C with respect to BC, CA, AB. The circum-circle ABC cuts $A'BC$ (say) in K (on $A'B$), M (on $A'C$) and AK, AM, AA' cut BC in P, R, Q, respectively. Prove that (1) the orthocenters of the associated triangles lie on circle ABC; (2) triangle AKM has its sides parallel to and equal twice the sides of the pedal triangle of ABC, and is also equal triangle formed by the above-named orthocenters; (3) $CP.a=b^2$, $BR.a=c^2$, $AP.a=AR.a=bc$, $BP.a=a^2-b^2$, $CR.a=a^2-c^2$, *i. e.*, $PR.a=2bc\cos A$, (4) hence BA touches circle ARC, which contains a Brocard-point of ABC; similarly for CA and circle APB; (5) $BR.CR'$, $AR''=abc=CP.BP'$, AP'' (where R', R'', P', P'' correspond to R, P, on CA, AB. respectively); K, K' are the Brocard constants ($k=a^2+b^2+c^2$) of ABB, $A'B'C'$; then $K'=K=\triangle^2/R^2$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1) Since the triangles $A'BC$, $B'AC$, $C'AB$ are equal to ABC, respectively, and A', B', C' are the images of A, B, C, the orthocenters of the triangles are the images of the orthocenter O, of the triangle ABC with respect to its sides.



But $BS.SE=AS.SC$, or $a\sin C.SE=a\cos C.c\cos A$. $\therefore$ $SE=c\cos A\cot C=SO$.

Similarly, $DQ=b\cos C\cot B=QO$, $TF=a\cos B\cot A=TO$.

$\therefore$ D, E, F are the orthocentres of the triangles.

(2) ArcKDC = arcCEA, both measured by $\angle B$.

ArcDC = arcCE, both measured by $\angle(\frac{1}{2}\pi-C)$.

$\therefore$ ArcKD = arcAE and DE is parallel to KA.

ArcMKB = arcBFA, both measured by $\angle C$.

ArcDB = arcBF, both measured by $\angle(\frac{1}{2}\pi-B)$.

$\therefore$ ArcDM = arcFA, and DF is parallel to AM.

$\therefore$ ArcKBA = arcDBF, arcDCE = arcMEA, arcKDM = arcFAE.

$\therefore$ $DF=KA=2QT$, $DE=MA=2QS$, $KM=FE=2TS$.

Also DE is parallel to AK is parallel to QS, DF is parallel to MA is parallel to QT, FE is parallel to TS. $\triangle DFE=\triangle AKM$ (three sides of one equal three sides of other).

(3) From triangle PAC, $\angle PAC=\angle B$, $\angle P=\angle A$.

$\therefore$ $CP\sin A=b\sin B$ or $CP.a=b^2$.

Similarly, from triangle BAR, $BR\sin A=c\sin C$, or $BR.a=c^2$, $\angle P=\angle R=\angle A$.

$\therefore AP=AR$ and $AP.a=AR.a$.

But $AP\sin A=b\sin C$ or $AP.a=bc=AR.a$.

$BP.a=(a-CP)a=a^2-CP.a=a^2-b^2$.

$CR.a=(a-BR)a=a^2-BR.a=a^2-c^2$.

$PR\sin A=AP\sin(\pi-2A)=2AP\sin A\cos A$.

$\therefore PR.a=2AP.a\cos A=2bc\cos A$.

(4) Since $BR.BC=AB^2$, AB touches the circle through ARC at A: therefore one of the Brocard points is on this circumference. Since $CP.CB=CA^2$, CA touches the circle through APB at A, which contains the other Brocard point.

(5) $BR=c^2/a$, $CR'=a^2/b$, $AR''=b^2/c$, $CP=b^2/a$, $BP'=a^2/c$, $AP''=c^2/b$.

$\therefore BR.CR'.AR''=CP.BP'.AP''=abc$; $(B'C')^2=c^2+b^2-2bc\cos 3A=a^2+8bc\cos A\sin^2 A$; $(A'C')^2=b^2+8ac\cos B\sin^2 B$, $(A'B')^2=c^2+8ab\cos C\sin^2 C$.

$$\therefore K'-K=8(bc\cos A\sin^2 A+ac\cos B\sin^2 B+ab\cos C\sin^2 C)$$
$$=32\triangle^2(\cos A/bc+\cos B/ac+\cos C/ab)$$
$$=(16\triangle^2/a^2b^2c^2)\Sigma(2a^2b^2-a^4)=256\triangle^4/a^2b^2c^2=16\triangle^2/R^2.$$

181. Proposed by **WILLIAM HOOVER**, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the extremities of the latera recta of all ellipses having a given major axis $2a$ lie on the parabola $x^2=-a(y-a)$.

Solution by **L. C. WALKER**, A. M., Petaluma High School, Petaluma, Cal.; **J. R. HITT**, Coral Institute, San Marcos, Tex.; and the **PROPOSER**.

If (x_1, y_1) be an extremity of one of the latera recta, plainly, $y_1=b^2/a$, or $b^2=ay_1$....(1); also, $a^2-b^2=a^2e^2=x_1^2$....(2), b and e having the usual meanings. Eliminating b from (1) and (2), $x_1^2=-a(y_1-a)$.

Also solved by *J. SCHEFFER*, and *G. B. M. ZERR*.

CALCULUS.

137. Proposed by **F. P. MATZ**, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Develop the equation of the curve assumed by the inextensible and revolving skipping rope.

No solution of this problem has been received.

138. Proposed by **M. E. GRABER**, A. B., Tutor in Mathematics, Heidelberg University, Tiffin, O.

Find the curve the length of whose arc measured from a given point is a mean proportional between the ordinate and twice the abscissa.

Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa., and the **PROPOSER**.

From the problem, $s^2=2xy$ or $s=\sqrt{(2xy)}$.

$$ds=\sqrt{(dx^2+dy^2)}=\frac{1}{\sqrt{2}}[\sqrt{(y/x)}dx+\sqrt{(x/y)}dy] \text{ or}$$

$$\sqrt{[1+(dy/dx)^2]}=\frac{1}{\sqrt{2}}[\sqrt{(y/x)}+\sqrt{(x/y)}(dy/dx)].$$

Let $y=mx$. $\therefore dy/dx=m+x(dm/dx)=m+px$.

$$\therefore \sqrt{[1+(m+px)^2]}=\frac{1}{\sqrt{2m}}(2m+px). \quad \frac{x^2(1-2m)p^2}{2m}+2x(1-m)p=(1-m)^2.$$

$$\therefore p=dm/dx=\frac{[1-m][1-\sqrt{(2m)}]\sqrt{(2m)}}{x[1-2m]}=\frac{\sqrt{[2m][1-m]}}{x[1+\sqrt{(2m)}]}.$$

$$\therefore dx/x=\frac{[1+\sqrt{(2m)}]\,dm}{[1-m]\sqrt{[2m]}}.$$

$$\therefore \log[Cx(1-m)]=\frac{1}{\sqrt{2}}\log\left[\frac{1+\sqrt{m}}{1-\sqrt{m}}\right].$$

$$\log[C(x-y)]=\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\right]=\frac{1}{\sqrt{2}}\log\left[\frac{x+y+2\sqrt{xy}}{x-y}\right].$$

For the given point $x=a$, $y=b$.

$$\therefore C=\frac{[a+b+2\sqrt{(ab)}]^{1/\sqrt{2}}}{[a-b][a-b]^{1/\sqrt{2}}}.$$

$$\therefore \frac{[x+y+2\sqrt{(xy)}]^{1/\sqrt{2}}}{[x-y][x-y]^{1/\sqrt{2}}}=\frac{[a+b+2\sqrt{(ab)}]^{1/\sqrt{2}}}{[a-b][a-b]^{1/\sqrt{2}}}.$$

$$\therefore \frac{a-b}{x-y}=\left[\frac{[x-y][a+b+2\sqrt{(ab)}]}{[a-b][x+y+2\sqrt{(xy)}]}\right]^{1/\sqrt{2}}, \text{ or } [y-x]^{1/\sqrt{2}}=c\,\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}.$$

139. Proposed by **WM. FRED FLEMING**, Chicago, Ill.

A tin watering-pot is constructed by joining the frustums of two right cones, so that their intersection is a mathematical one, their axes meeting at an angle of 45°. The bases of the smaller frustum are 2 inches and 4 inches in diameter, its altitude 8 inches. The bases of the larger frustum are 10 inches and 12 inches in diameter, its altitude 15 inches. In joining the two frustums the edges of the two larger bases are brought into coincidence. Water is poured into the vessel until it begins to run out of the spout. How many gallons (231 cubic inches) are required? How much water is in the spout and how much in the can? The vessel is tilted forward (in the plane of the axes of the two frustums) sufficiently to allow one-half of the water to run out. How much of the liquid is left in the spout and can, and what is the area of the surface of the water in spout and can? Through what angle has the vessel been tilted?

No solution of this problem has been received.

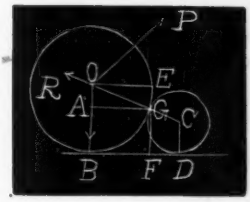
MECHANICS.

135. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

What force acting at an inclination ω with a horizontal line on the center of a wheel of given weight will roll the wheel over an immovable cylindric log whose diameter is $(1/m)$th that of the wheel?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $CD=GC=a$, $OB=OE=ma$, P=force, R=reaction, W=weight of wheel, $\angle POE=\omega$, $\angle AOC=\theta$.



Resolving vertically, $W=R\cos\theta$.

Resolving horizontally, $P\cos\omega=R\sin\theta$.

$\therefore P\cos\omega/\sin\theta=W/\cos\theta$.

$\therefore P=W\tan\theta\sec\omega$.

$GF=a+a\cos\theta=a[1+\cos\theta]$.

$AO=ma\cos\theta=ma-a[1+\cos\theta]$.

$\therefore \cos\theta=[m-1]/[m+1]$. $\tan\theta=2\sqrt{m}/[m-1]$.

$$\therefore P=\frac{2W\sqrt{(m)}\sec\omega}{m-1}.$$

136. Proposed by F. T. WRIGHT, Ph. B., Schenectady, N. Y.

In an air brake test a train moving at 22 miles an hour on a down grade of one per cent. was stopped in 91 feet. There was 94 per cent. of the train braked. Taking the fractional resistance as 8 pounds per ton, find the net brake resistance per ton.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let the train weight T tons of 2240 lbs. The work due to gravity is $T(91\times2240)/100$. 22 miles per hour$=32\frac{4}{15}$ feet per second.

Let x=net brake resistance, $g=32.16$. Then

$$\frac{2240(32\frac{4}{15})^2T}{64.32}=8\times91T+\frac{94\times91Tx}{100}-\frac{91\times2240\,T}{100}.$$

$\therefore 36258.53=728+85.54x-2038.40$. $85.54x=37568.93$.

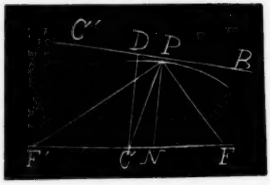
$x=439.2$ pounds per ton.

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Professeor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A uniform inextensible string rests against the inner side of a smooth elliptic wire semi-axes a and b, and is repelled from the foci and the center by the following forces: μ/rd and $\nu/r'd$ emanating from the foci, and $\pi c/d$ from the center, the distances of any point on the string from the foci being r and r', respectively, its distance from the center being c, and the semi-conjugate diameter corresponding to the point being d. Find the pressure on the wire at any point.

Solution by the PROPOSER.

Let P be any point, $C'B$ the tangent, PN the normal at P. $PF=r$, $PF'=r'$, $CP=c$, $\angle CPC'=\varphi$, $\angle FPB=\angle F'PC'=\theta$, $CD=p$, T=tension, n=normal force, m=tangential force, ρ=radius of curvature, R=pressure on wire at P. The equations of equilibrium are



$$dT/ds+m=0,\quad T/\rho-R=n.$$

But $m=(\mu/rd)\cos\theta+(\nu/r'd)\cos\theta+(\pi c/d)\cos\varphi$.

$$n=\left(\frac{\mu}{r}+\frac{\nu}{r'}\right)\frac{\sin\theta}{d}+\frac{\pi c}{d}\sin\varphi.$$

But $\sin\theta=b/d$, $\sin\varphi=ab/dc$, $\cos\theta ds=dr=dr'$, $\cos\varphi ds=/dc$.

$$\therefore\ dT+\frac{\mu}{rd}dr+\frac{\nu}{r'd}dr'+\frac{\pi c}{d}dc=0....(1),$$

$$T/\rho-R=\left(\frac{\mu}{r}+\frac{\nu}{r'}\right)\frac{b}{d^2}+\frac{\pi ab}{d^2}....(2)$$

are the equations of equilibrium.

Now $d=\sqrt{(rr')}=\sqrt{[r(2a-r')]}=\sqrt{[r'(2a-r)]}=\sqrt{[a^2+b^2-c^2]}$.

$$\therefore\ dT+\frac{\mu dr}{r\sqrt{[r(2a-r)]}}+\frac{\nu dr'}{r'\sqrt{[r'(2a-r')]}}+\frac{\pi cdc}{\sqrt{[a^2+b^2-c^2]}}=0.$$

Integrating, $T+\frac{\mu\sqrt{[2a-r]}}{a\sqrt{r}}-\frac{\nu\sqrt{[2a-r']}}{a\sqrt{r'}}-\pi\sqrt{[a^2+b^2-c^2]}=C$,

$$\text{or } T-\frac{\mu}{a}\sqrt{\left(\frac{r'}{r}\right)}-\frac{\nu}{a}\sqrt{\left(\frac{r}{r'}\right)}-\pi d=C.$$

Let T_0 be the tension of the string at the extremity of the minor axis. Then $r=r'=a$, $d=a$.

$$\therefore\ C=\frac{aT_0-\mu-\nu-\pi a^2}{a}.$$

$$\therefore\ T=\frac{\mu}{a}\sqrt{\left(\frac{r'}{r}\right)}+\frac{\nu}{a}\sqrt{\left(\frac{r}{r'}\right)}+\pi d+\frac{aT_0-\mu-\nu-\pi a^2}{a}.$$

But $\rho=a^2b^2/p^3$ and $p=ab/d$.

$$\therefore\ 1/\rho=p^3/a^2b^2=ab/d^3=ab/\sqrt{[r^3r'^3]}.$$

$$\therefore\ T/\rho=\frac{b}{r^2r'^2}(\mu r'+\nu r+\pi arr')+\frac{aT_0-\mu-\nu-\pi a^2}{a\rho}.$$

$$\therefore\ \frac{b}{r^2r'^2}(\mu r'+\nu r+\pi arr')+\frac{aT_0-\mu-\nu-\pi a^2}{a\rho}-R=\frac{b}{r^2r'^2}(\mu r'+\nu r+\pi arr').$$

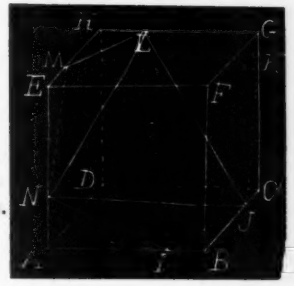
$$\therefore\ R=\frac{aT_0-\mu-\nu-\pi a^2}{a\rho}.$$

AVERAGE AND PROBABILITY.

113. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

A given cube is cut by a plane in such a manner that the lines of section form a regular hexagon. What is the mean area of this hexagon?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.



Let $ABCD-H$ be the cube, side a.

$HM=EN=AI=BJ=CK=GL=x$.

$LH=EM=AN=BI=CJ=GK=a-x$.

Then $IJKLMN$ is a hexagon.

$IJ=JK=KL=LM=MN=NI=\frac{1}{2}\sqrt{2}\,(a^2+2x^2-2ax)$.

$IK=JL=KM=LN=MI=NJ=\frac{1}{2}\sqrt{2}\,(2a^2+2x^2-2ax)$.

$\therefore$ The hexagon is a regular hexagon.

Area of this hexagon $=\frac{3}{2}(a^2+2x^2-2ax)\frac{1}{2}\sqrt{3}=u$.

$$\text{Average area}=\Delta=\int_{\frac{1}{2}a}^{a}u\,dx\Big/\int_{\frac{1}{2}a}^{a}dx.$$

$$\therefore\ \Delta=\frac{3\frac{1}{2}\sqrt{3}}{a}\int_{\frac{1}{2}a}^{a}(a^2+2x^2-2ax)dx=a^2\tfrac{1}{2}\sqrt{3}.$$

114. Proposed by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

If a regular polygon of n sides be placed at random on another equal polygon, show that the chance that the center of the first will fall on the second polygon is $\dfrac{\pi}{2[\pi+n\tan(\pi/n)]}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let AOB, DCE be a triangle of each polygon. Let DCE move parallel to itself, then the center D will trace out the polygon of $2n$ sides $KLMGN$, etc.

Let $BO=CD=r$, $\angle BCE=\theta=$the angle BO makes with DC. Then the perpendicular distance from AB to $HG=r\cos[(\pi/n)-\theta]$, $HA=r\sec(\pi/n)\cos[(\pi/n)-\theta]=r[\cos\theta+\sin\theta\tan(\pi/n)]$, $OH=r[1+\cos\theta+\sin\theta\tan(\pi/n)]$.

$MG=\frac{1}{2}\sqrt{(CD^2+BG^2-2CD.BG\cos\theta)}=r\sin\theta\sec(\pi/n)$.

$HG=2r[1+\cos\theta+\sin\theta\tan(\pi/n)]\sin(\pi/n)$.

$HL=HG-LG=r[2\cos\theta\sin(\pi/n)+2\sin\theta\sin(\pi/n)\tan(\pi/n)-\sin\theta\sec(\pi/n)]$.

Area $HOG=\frac{1}{2}r^2[1+\cos\theta+\sin\theta\tan(\pi/n)]^2\sin(2\pi/n)$.

Area $MGN=\frac{1}{2}r^2[2\sin\theta\cos\theta\tan(\pi/n)+\sin^2\theta\tan^2(\pi/n)-\sin^2\theta]\sin^2(\pi/n)$.

The total number of positions is $n(\triangle OGH-\triangle MGN)=nr^2\sin(2\pi/n)[1+\cos\theta+\sin\theta\tan(\pi/n)]=A$. The number of favorable positions is $n\triangle AOB=B$.

$\therefore\ B=\frac{1}{2}nr^2\sin(2\pi/n)$.

$$\therefore\ p=\int_0^{\pi/n}\beta d\theta\Big/\int_0^{\pi/n}A d\theta=\tfrac{1}{2}\int_0^{\pi/n}d\theta\Big/\int_0^{\pi/n}[1+\cos\theta+\sin\theta\tan(\pi/n)]d\theta=\frac{\pi}{2[\pi+n\tan(\pi/n)]}$$

Cor. When $n=\infty$, $p=\frac{1}{4}$; when $n=3$, $p=\pi/[2(\pi+3\frac{1}{2}\sqrt{3})]$; when $\pi=4$, $p=\pi/[2(\pi+4)]$; when $n=6$, $p=\pi/[2(\pi+2\frac{1}{2}\sqrt{3})]$.

BOOKS AND PERIODICALS.

Science of Mechanics. A critical and historical account of its development. By Dr. Ernst Mach, Professor of the History and Theory of Inductive Science in the University of Vienna. Translated from the German by Thomas J. McCormack. Second revised and enlarged edition, with 259 cuts and illustrations. 8vo. Cloth, xx+605 pages. Price, $2.00 net. Chicago: The Open Court Publishing Co.

This work differs from others on the same subject in treating of the principles of mechanics principally under the aspect of their development. To the student reading the subject for the first time, this method will prove easy and interesting. He will be able to master many principles at first reading that would require much study to master in other works. As a history of mechanics, it is excellent, and as a text-book on Mechanics it is admirable, its treatment being lucid, clear and forcible. It should be in the hands of every teacher of Physics and Mechanics.

Elementary Principles in Statical Mechanics. Developed with especial reference to the Rational Foundation of Thermodynamics. By J. Willard Gibbs, Ph. D., LL. D., Professor of Mathematical Physics in Yale University. 8vo. Cloth. xii+207 pages. Price, $4.00 net. New York: Charles Scribner's Sons.

The matter of this volume, the author says, consists in large measure of resulst which have been obtained by the investigations of Clausius, Maxwell, and Boltzmann, although the point of view and the arrangement is different. The first chapter considers the principle of conservation of extension-in-phase and derives what may be called the fundamental equation of statical mechanics. In the general case, the fundamental equation admits of integration, which gives a principle variously expressed, according to the point of view from which it is regarded, as the conservation of density-in-phase, or of extension-in-phase, or of probability-in-phase. This principle of the conservation of probability of phase is applied to the theory of errors in the calculated phase of a system, when the determination of the arbitrary constants of the integral equations are subject to error. In the third chapter, the principle of conservation of extension-in-phase is applied to the integration of the differential equation of motion. In the fourth and following chapters the author returns to the consideration of statical equilibrium and confines his attention to the conservative systems. In the fourteenth chapter thermodynamic is discussed, and in chapter fifteen is considered the modification of the preceding results which is necessary when systems composed of a number of entirely similar particles are considered.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

Among the contributed articles in the *Review of Reviews* for June are the following: "Bowdoin College: a Century of Service," by William I. Cole; "The Queen Regent and the Young King of Spain," by Helene Vacaresco; and "Two American Novelists" (Bret Harte and Frank Stockton).

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and Published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

CRISTOFORO ALASIA.

THE AMERICAN MATHEMATICAL MONTHLY.

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BIOGRAPHY.

CRISTOFORO ALASIA.

By DR. GEORGE BRUCE HALSTED, Austin, Texas.

When a new star comes out in the skies, thither turns the observing eye. "Le Matematiche" is a new luminary among scientific periodicals, though sailing safely now far into its second year.

Its director, its creator, Professor Alasia, has won the confidence and is attracting the attention of the mathematical world.

A sketch of his career cannot but be opportune, however brief and inadequate. That Professor Alasia is still very young to have won so prominent a position will be seen when we say, he was born in 1869, in Sassari, Sardinia.

His university course was carried on by turns in three different cities, Turin, Cagliari, and Rome. At Turin he was so fortunate as to have for masters those extraordinarily influential men, D'Ovidio and Peano.

At Rome he completed two courses at the School of Application for engineers under the direction of Senator Cremona. The course there in rational mechanics under Professor Cerruti is justly famous.

The sudden death of young Alasia's father recalled him to Sardinia. He won his first teaching position by competition. On this occasion he published his first book, on the theory of equations, (Naples, 1893).

His love of science did not however prevent him from also occupying himself with the fine arts; he has won two prizes at expositions of dilettanti in painting. Professor Alasia is a gifted linguist. He has an elegant Italian style, writes the purest French, and at present is engaged with Professor Dionisio Gambioli in translating into Italian Cajori's History of Physics. They will enlarge the work by two additional chapters and copious notes. It is expected to appear at the end of this year.

In his Essay on the nomenclature (bibliographic) of the New Geometry of the triangle, our author proposes to give the most complete possible list of the terms which have entered the domain of geometry in these latter years, to give their veritable signification, to investigate what geometer has first used them, upon what occasion, etc.

In addition, when it is a question of a point or a straight, he has given its representation according to the method of Grassmann; and for circles and conics he has given the equation in barycentric or normal coördinates.

The monthly journal of pure and applied mathematics founded by Professor Alasia, "Le Matematiche," has had an extraordinary success.

The last thing ever written by the great Hermite was for it. Professor Alasia has been able to win the support and friendship of many of the most illustrious of living mathematicians, for example, Poincaré.

His is a charming figure in the new renaissance of creative productivity in Italy. His fine judgment and powers of assimilation are illustrated in his Poligeometrognomia generale e la Geometria Non-Euclidea del Chrystal, a translation of Chrystal's Non-Euclidean Geometry, preceded by a general resumé, historic and bibliographic, in exposition of the foundations of geometry, remarkable in erudition and breadth of insight.

His splendid fertility is amply shown in the subjoined list of his other writings:

Elementi della Teoria delle equagioni, ecc.—Napoli, 1893, B. Pellermo, ed.

Sulla deviazione dei gravi,—lettera alla Societa Astronomica d'Francia.

Su di alcune proprieta dei numeri e delle congruenze, ecc. Civitanova—Marche, 1898. C'est le vol. VI, sect. Science, de la Collection d'Audes italiane qui a été publiée par l'éditeur D. Natalucci.

Sulle involuzioni di ordini superiori,—Civitanova, 1898, Natalucci, ed.

Su di alcune proprieta delle linee geodetiche,—Sassari, 1898.

Esercize ed applicazioni di Calcolo infinitesimale e íntegrale—Citta di Castello, 1898, S. Lapi, ed.

Calcolo Grafico ed applicazioni alla Statica, ibid. 1899.

La Recente Geometria del triangolo, ibid. 1900.

566 Relazioni metriche e trigonometriche fra gli elementi d'un triangolo piano, —ibid. 1900.

Geometria e Trigonometria della Sfera, Milano, 1900, U. Hoepli, ed.

Esercizi ed applicazioni di Trigonometria piana, ibid. 1901.

Su di alcuni teoremi di Le Paige e Deruits, Madrid, 1899.